AN ANALYSIS OF THE EFFECTS OF CHANGES IN SAVINGS RATES USING NEOCLASSICAL CGE MODELS: LESSONS FROM ANALYTICAL SOLUTIONS TO SIMPLE TWO-SECTOR MODELS WITH ENDOGENOUS INTEREST RATES

Serdar Sayan[*]
No. 94-11

April 1994

[*] Department of Economics
Bilkent University
06800 Ankara, Turkey

Phone: +90(312)290-2071 **Fax:** +90(312)266-5140
E-Mail: sayan@bilkent.edu.tr
Web: http://www.bılkent.edu.tr/~sayan

AN ANALYSIS OF THE EFFECTS OF CHANGES IN SAVINGS RATES USING NEOCLASSICAL CGE MODELS: LESSONS FROM ANALYTICAL SOLUTIONS TO SIMPLE TWO-SECTOR MODELS WITH ENDOGENOUS INTEREST RATES [*]

Traditional Computable General Equilibrium (CGE) models do not have explicitly modeled money markets as they take money as neutral consistently with the Walrasian framework. Due to the absence of money markets, these models typically omit the formulation of an interest rate variable. It is argued in the paper that this omission affects the usefulness of the CGE framework negatively, especially in the analysis of the general equilibrium impact of macro shocks. A formulation is suggested here to allow for an endogenous determination of interest rate within a simple, Walrasian general equilibrium model that is analytically solvable. The step-by-step development of the model starting from simpler models includes relevant features of the traditional models in the CGE literature.

The comparative static effects of an exogenous increase in the propensity to save coefficient are investigated to observe the behavior of formulated interest rate variable in response to shocks. The implications of results are discussed. In particular, the directional uncertainty concerning the comparative static effect on the interest rate is shown to be resolved by considering the factor intensities of production in different sectors.

Key Words: Interest Rates, Walrasian Models, Computable General Equilibrium Modeling, Savings Rates.

* The author expresses thanks to Richard Carlson for stylistic comments, and to Pelin Kale for proofreading and rechecking the derivations, without implication for any errors that might have remained.

Introduction

Increasing degree of non-linearity and the number of endogenous variables/equations makes it difficult, if not impossible, to find analytical solutions to systems of non-linear equations. A CGE model can be viewed, in purely mathematical terms, as a simultaneous system of non-linear equations for which it is *impossible* to find an analytical solution. Solution of these models, therefore, requires the use of numerical techniques. Advances in computing technology and the introduction of canned software capable of providing numerical solutions for large non-linear systems have substantially increased the popularity of CGE models in the last decade or so. Nonetheless, these models remain "puzzling black boxes," as B. Bolnick once put it, for large numbers of students and practitioners. Constructing analytically solvable models that mimic certain features of large applied models would not only help one better understand the structure of larger models but also facilitate an understanding of the underlying adjustment mechanisms to exogenous shocks.

This paper first presents an analytical solution to a very simple, 2x2 general equilibrium model in the Walrasian tradition under the assumption that there is no savings in the economy. The 2x2 model is then modified to allow for savings. The modified model incorporates investment behavior using a *capital composition matrix* so as to mimic the treatment of investment in traditional, static CGE models in the literature. It is shown, within the context of this model, how the solution, once obtained, can be used for an analytical investigation of the comparative-statics effects of an exogenous change in the *marginal propensity to save* (mps) parameter. The purpose of this exercise is both to illustrate the simulation experimentation with CGE models and to lay the groundwork for a discussion of the treatment of interest rates in the applied literature.

It is suggested in the paper that despite the Walrasian neutrality of money, an interest rate variable can be introduced to the model under certain assumptions. In particular, the assumption of the perfect mobility of capital between sectors can be taken to imply the existence of a perfect market in existing capital stock. In turn, the existence of such a market can be viewed as a substitute for a money market not incorporated into the model as the (durable) capital goods now act as stores of value --thereby replacing money as a store of value. Based on this line of reasoning, a formulation allowing for the endogenous determination of an interest rate is added to model equations. The comparative-statics experimentation for an exogenous change in the *mps* is repeated with the model version that includes the interest rate formulation, and results are discussed.

A Simple Walrasian General Equilibrium Model

Consider the following simple general equilibrium model for an economy with no government and no foreign trade. The Greek characters in the equations (1) through (13) represent parameters; small case letters and overlined capital letters stand for exogenous variables, and the rest of the symbols are endogenous variables.

$$X_1 = L_1^{\alpha} \cdot K_1^{(1-\alpha)} \qquad \textbf{(1)}$$

$$X_2 = L_2^{\beta} \cdot K_2^{(1-\beta)} \qquad \textbf{(2)}$$

$$W = \alpha . P_1 \cdot L_1^{(\alpha-1)} \cdot K_1^{(1-\alpha)} \qquad \textbf{(3)}$$

$$R = (1-\alpha) . P_1 \cdot L_1^{\alpha} \cdot K_1^{-\alpha} \qquad \textbf{(4)}$$

$$W = \beta . P_2 \cdot L_2^{(\beta-1)} \cdot K_2^{(1-\beta)} \qquad \textbf{(5)}$$

$$R = (1-\beta) . P_2 \cdot L_2^{\beta} \cdot K_2^{-\beta} \qquad \textbf{(6)}$$

$$\overline{L} = L_1 + L_2 \qquad \textbf{(7)}$$

$$\overline{K} = K_1 + K_2 \qquad \textbf{(8)}$$

$$C_1 = \theta . \frac{Y}{P_1} \qquad \textbf{(9)}$$

$$C_2 = (1-\theta) . \frac{Y}{P_2} \qquad \textbf{(10)}$$

$$Y = W.\overline{L} + R.\overline{K} \qquad \textbf{(11)}$$

$$X_1 = C_1 \qquad \textbf{(12)}$$

$$X_2 = C_2 \qquad \textbf{(13)}$$

In the economy, two commodities, X_1 and X_2, are produced by perfectly competitive firms which employ Cobb-Douglas type CRS technologies. Equations (1) and (2) represent the typical firm's production functions. Each factor of production employed in sector i ($i \varepsilon \{1,2\}$) is paid (at the nominal rates of W for labor, and R for capital) according to its marginal productivity leading to factor demand functions in equations (3) to (6) where P_i is the (competitive) price of commodity i. The economy-wide supply of capital and labor is fixed at $\overline{K}$ and $\overline{L}$, respectively [equations (7) and (8)], but factors, including capital, are ***perfectly mobile*** between sectors.

On the demand side, a single representative household owning all factors of production tries to maximize its utility -- given by a Cobb-Douglas type utility function -- by spending *all* its factor income [equation (11)] on consumption. The problem of the household trying to determine the amounts to be consumed of each commodity, C_i , can be formulated as

$$\begin{array}{ll} Max & U(C_1, C_2) = C_1^{\theta}.C_2^{(1-\theta)} \\ subject\ to & P_1.C_1 + P_2.C_2 = Y \end{array}$$

the solution of which yields the Marshallian demand functions in (9) and (10). Finally, the commodity market equilibrium conditions are represented by equations (12) and (13).

Walras' Law and Its Implications

A simple counting of equations and endogenous variables reveals that the system is square --with as many equations as endogenous variables (W, R, Y, X_i, L_i, K_i, C_i, P_i with $i\varepsilon\{1,2\}$). By Walras' Law,[1] however, the number of *independent* equations is one fewer than the number of endogenous variables. To prove that Walras' Law holds in the model, one can manipulate equations (1) through (8) to show that

$$P_1.X_1 + P_2.X_2 = W.\ \overline{L} + R.\ \overline{K}.$$

But the RHS of this equation is equal to Y by (11) so that we can write $P_1.X_1+P_2.X_2=Y$ which is the budget constraint behind the utility maximizing Marshallian demands. But the demand functions (9) and (10) imply $P_1.C_1 + P_2.C_2 = Y$ which, by (12) and (13), is the same as $P_1.X_1 + P_2.X_2 = Y$. Hence, one equation in the system is redundant since equations (1) through (8) plus (11) yield the same equation as equations (9) and (10) plus (12) and (13). That the system satisfies Walras' Law can also be proven by showing first that equilibrium in the first commodity market implies, by (8) and (12), that

$$P_1 . X_1 = \theta . Y .$$

Substituting for Y from (11), and using (7) and (8), we get

$$P_1 . X_1 = \theta . [(W . L_1 + R . K_1) + (W . L_2 + R . K_2)] .$$

[1] By Walras' Law, in a system such as the simple model here, equilibrium in (N-1) of the markets necessarily requires that the N^{th} market is also in equilibrium. This implies that one of the equations is redundant and that the system can only be solved for relative prices but not for each sectoral price level.

But the first term in brackets is equal to $P_1.X_1$ by (1), (3) and (4); and the second to $P_2.X_2$ by (2), (5) and (6). Hence, we write

$$P_1 \cdot X_1 = \theta.(P_1 \cdot X_1 + P_2 \cdot X_2).$$

It follows now from (10) that $P_2.C_2 = (1\text{-}\theta).Y$, or equivalently

$$P_2 \cdot C_2 = (1-\theta).(P_1 \cdot X_1 + P_2 \cdot X_2).$$

Summing up the last two equations, we get

$$P_1 \cdot X_1 + P_2 \cdot C_2 = [\theta + (1-\theta)].(P_1 \cdot X_1 + P_2 \cdot X_2)$$

which is equivalent to $C_2 = X_2$. Therefore, $X_1 = C_1 \leftrightarrow X_2 = C_2$ as Walras' Law states.

With Walras' Law proven to hold in the model, the number of *independent* equations is observed to be one fewer than the number of endogenous variables. As the equality of the number of independent equations and that of endogenous variables is a *necessary* condition for obtaining a solution, this leaves essentially three options before the model builder trying to solve the model. First, one can choose one of the commodities/factors as the *numerairé* and express other nominal variables in terms of the (quantities of) chosen commodity/factor. This amounts to fixing one of the commodity/factor prices and makes the number of independent equations equal to that of endogenous variables. The second option is the addition of an equation defining a price index whose value is constant. That is,

$$\omega_1.P_1 + \omega_2.P_2 = \overline{P}$$

where ω_i's are exogenous weight factors and $\overline{P}$ is the constant price index, i.e., no-inflation benchmark. This approach increases the number of equations by one without adding an endogenous variable thereby making the number of independent equations equal to that of endogenous variables. Finally, after a slight modification, the model can be solved for relative prices and for real (quantity) variables.

Under this third option, the nominal income variable, Y, needs to be converted into a real income variable. But since there are two commodities, real income must be defined in terms of both (the purchasing power over) commodity 1, i.e., Y/P_1; and (the purchasing power over) commodity 2, i.e., Y/P_2. This treatment increases the number of endogenous variables by one --as we now have both Y/P_1 and Y/P_2 instead of Y alone. To accommodate this increase in the number of endogenous variables, we replace equation (11) with

$$\frac{Y}{P_1} = \frac{W}{P_1}.\overline{L} + \frac{R}{P_1}.\overline{K} \tag{11a}$$

$$\frac{Y}{P_2} = \frac{W}{P_2}.\overline{L} + \frac{R}{P_2}.\overline{K} \tag{11b}$$

The resulting system has now 14 equations [equations (1) through (10), (11a), (11b), (12) and (13)] in 14 endogenous variables: 4 relative price variables (i.e., W/P_i and R/P_i) and 10 real (quantity) variables (i.e., X_i, L_i, K_i, and Y/P_i). The selection of this last option is more directly consistent with the best-known implication of Walras' Law: a Walrasian system can determine the relative prices but not the absolute price levels separately.

The Analytical Solution

To solve the system of equations (1) through (13), we write an equivalent but downsized formulation as follows:

$$\alpha.P_1.L_1^{(\alpha-1)}\cdot K_1^{(1-\alpha)} = \beta.P_2\cdot L_2^{(\beta-1)}\cdot K_2^{(1-\beta)} \tag{14}$$

$$(1-\alpha).P_1.L_1^{\alpha}\cdot K_1^{-\alpha} = (1-\beta).P_2\cdot L_2^{\beta}\cdot K_2^{-\beta} \tag{15}$$

$$\overline{L} = L_1 + L_2 \tag{16}$$

$$\overline{K} = K_1 + K_2 \tag{17}$$

$$\frac{L_1^{\alpha}\cdot K_1^{(1-\alpha)}}{L_2^{\beta}\cdot K_2^{(1-\beta)}} = \overline{\theta}.\frac{P_2}{P_1} \tag{18}$$

where $\overline{\theta} = \theta/(1-\theta)$. Solving the system of equations (14) through (18) --five equations in five endogenous variables (i.e., L_i, K_i and P_2/P_1), we obtain the reduced forms for endogenous variables L_i and K_i:

$$L_1 = \frac{\alpha.\overline{\theta}}{(\alpha.\overline{\theta} + \beta)}.\overline{L} \tag{19a}$$

$$L_2 = \frac{\beta}{(\alpha.\overline{\theta} + \beta)}.\overline{L} \tag{19b}$$

$$K_1 = \frac{(1-\alpha).\overline{\theta}}{(1-\alpha).\overline{\theta} + (1-\beta)}.\overline{K} \tag{20a}$$

$$K_2 = \frac{(1-\beta)}{(1-\alpha).\overline{\theta} + (1-\beta)}.\overline{K}\ . \tag{20b}$$

Having found the solution values for L_i and K_i, we can solve for P_2/P_1 using (20). The values found can now be used to determine other relative prices (real returns to factors) -- W/P_i from (3) and (5), and R/P_i from (4) and (6); as well as real variables -- X_i from (1) and (2), and Y/P_i from (11a) and (11b).[2]

The Extended Model with Savings

It is now time to introduce savings into the model. To do this, we add the assumption that sector 1 is producing a "Final Consumption Good" alone, whereas the (aggregated) output of sector 2 can be used for both "Final Consumption" and as an "Investment Good."[3] Assuming further that total savings is a fixed fraction, *mps*, of income in the economy, we can write the following version of the system of equations (1) through (13).

$$X_1 = L_1^{\alpha} \cdot K_1^{(1-\alpha)} \quad \textbf{(21)}$$

$$X_2 = L_2^{\beta} \cdot K_2^{(1-\beta)} \quad \textbf{(22)}$$

$$W = \alpha . P_1 \cdot L_1^{(\alpha-1)} \cdot K_1^{(1-\alpha)} \quad \textbf{(23)}$$

$$R = (1-\alpha) . P_1 \cdot L_1^{\alpha} \cdot K_1^{-\alpha} \quad \textbf{(24)}$$

$$W = \beta . P_2 \cdot L_2^{(\beta-1)} \cdot K_2^{(1-\beta)} \quad \textbf{(25)}$$

$$R = (1-\beta) . P_2 \cdot L_2^{\beta} \cdot K_2^{-\beta} \quad \textbf{(26)}$$

$$\overline{L} = L_1 + L_2 \quad \textbf{(27)}$$

$$\overline{K} = K_1 + K_2 \quad \textbf{(28)}$$

$$C_1 = \gamma_1 \cdot \frac{(1-mps).Y}{P_1} \quad \textbf{(29)}$$

$$C_2 = \gamma_2 \cdot \frac{(1-mps).Y}{P_2} \quad \textbf{(30)}$$

$$Y = W.\overline{L} + R.\overline{K} \quad \textbf{(31)}$$

$$X_1 = C_1 \quad \textbf{(32)}$$

$$X_2 = C_2 + I_2 \quad \textbf{(33)}$$

---

[2] Note that the solution value for nominal income variable, Y, still can not be found while it is possible to find the real incomes expressed in terms of (the purchasing power over) commodity 1, Y/P_1; and commodity 2, Y/P_2; using (11a) and (11b), respectively.

[3] Even without reference to the aggregation issue, we can cite the usual example of *corn* whose output can both be used for final consumption and as an investment good (i.e., seed), to such a double-use commodity.

The system proven to satisfy Walras Law is a special case with mps=0 and I_2=0 of the latter system. Hence, it can be shown that this system of equations also satisfies Walras' Law.[4] I_2 here is the component of sector 2's output devoted to investment --as opposed to consumption. To be able to determine I_2 endogenously, we need to *add* the following equations to the system (21) through (33):[5]

$$I_2 = a_{21} \cdot DK_1 + a_{22} \cdot DK_2 \tag{34}$$

$$DK_1 = m_1 \cdot \frac{mps.Y}{PK_1} \tag{35}$$

$$DK_2 = m_2 \cdot \frac{mps.Y}{PK_2} \tag{36}$$

$$PK_1 = a_{21} \cdot P_2 \tag{37}$$

$$PK_2 = a_{22} \cdot P_2 \tag{38}$$

where DK_i: investment by sector of destination,
PK_i: cost of increasing capital stock of sector i by one unit (the price paid by sector i for each unit of investment good purchased from sector 2)
m_i : investment share parameters ($\Sigma m_i = 1$), and
a_{2j} : the second row elements of "capital composition matrix."[6]

The system (21) through (38) has 18 equations (17 of which are independent) and 18 endogenous variables: X_i, L_i, K_i, W, R, Y, P_i, C_i, I_2, DK_i and PK_i. To make the number of independent equations equal to the number of endogenous variables so as to be able to solve the system, we need to choose a numerairé or define an aggregate price index as discussed earlier. We choose commodity 1 as the numerairé and set P_1=1 to simplify expressions. We now downsize the system (21) through (38) -- in a similar way to what was done for the simpler model, with P_1=1 and write

[4] Note that consumption function LES coefficients, θ and (1-θ), in the former system have now been replaced by γ_i but this is not a fundamental difference since $\gamma_1=1-\gamma_2$.

[5] For details of this approach to formulation of investment demand, see. Derviş, de Melo and Robinson; or Robinson, Kilkenny and Hanson.

[6] Note that a_{ij} is the amount of sector i's output required to increase sector j's capital stock by one unit. $a_{11}=a_{21}=0$ since sector 1 does not produce any investment goods.

$$\alpha . L_1^{(\alpha-1)} \cdot K_1^{(1-\alpha)} = \beta . P_2 \cdot L_2^{(\beta-1)} \cdot K_2^{(1-\beta)} \quad \textbf{(39)}$$

$$(1-\alpha) . L_1^{\alpha} \cdot K_1^{-\alpha} = (1-\beta) . P_2 \cdot L_2^{\beta} \cdot K_2^{-\beta} \quad \textbf{(40)}$$

$$\overline{L} = L_1 + L_2 \quad \textbf{(41)}$$

$$\overline{K} = K_1 + K_2 \quad \textbf{(42)}$$

$$\frac{L_1^{\alpha} \cdot K_1^{(1-\alpha)}}{L_2^{\beta} \cdot K_2^{(1-\beta)}} = \overline{\theta} . P_2 \quad \textbf{(43)}$$

Note that the system (39) through (43) is a special case of (14) through (18) with P_1=1 and $\overline{\theta}=\gamma_1$(1-mps)/[1-$\gamma_1$(1-mps)].[7] The latter system of 5 equations in 5 endogenous variables (L_i, K_i and P_2) gives the same solution for L_i and K_i as in equations (19a) through (20b), except that θ now is given by γ_1(1-mps). One can proceed to solve for P_2 by substituting RHSs of (19a) through (20b) in (43). The values obtained for P_2, L_i and K_i can then be used to solve for other variables using equations (21)-(38) as explained earlier.

The Analysis of the Effects of a Change in "*mps*"

CGE models solved by employing numerical solution techniques are used most commonly for simulation experiments designed to investigate the *comparative statics* effects of a change in one or more of the parameters/exogenous variables on (base period/ benchmark) equilibrium values of the endogenous variables. An "analytical equivalent" of such an investigation can be repeated using the system (21) through (38) to see the effects of changes in *mps*, an exogenous variable.

To analyze the effects of, say, an increase in *mps*, i.e., **d**mps>0, we first substitute the solution values for L_i and K_i in (43) and rewrite the equation after rearranging terms as

$$\frac{\alpha^{\alpha} . (1-\alpha)^{(1-\alpha)}}{\beta^{\beta} . (1-\beta)^{(1-\beta)}} \cdot \left[\frac{\overline{L}}{(\alpha\overline{\theta}+\beta)} \right]^{(\alpha-\beta)} \cdot \left[\frac{\overline{K}}{(1-\alpha).\overline{\theta}+(1-\beta)} \right]^{(\beta-\alpha)} = P_2 \ . \quad \textbf{(43a)}$$

[7] Though treating P_2/P_1 as a single variable is not fundamentally different than choosing commodity 1 as the numeraire and setting P_1=1, the latter option is chosen to simplify expressions.

Taking the natural log of both sides and totally differentiating --with (19a) and (20a) in mind, we get

$$(\alpha - \beta)\,[\frac{K_1}{\overline{K}} - \frac{L_1}{\overline{L}}].\frac{d\overline{\theta}}{\overline{\theta}} \;=\; p_2 \tag{43b}$$

where p_2=**d**ln P_2=**d**P_2/P_2. We can use (27) and (28) to rewrite (43b) as

$$\frac{(\alpha - \beta)}{\overline{K}.\overline{L}}.(K_1 \cdot L_2 - K_2 \cdot L_1).\frac{d\overline{\theta}}{\overline{\theta}} \;=\; p_2 \;. \tag{43c}$$

The signs of the first two terms on the left hand side of (43c) are unknown. However, (39) and (40) above imply that

$$K_1 \cdot L_2 - K_2 \cdot L_1 = \frac{\beta}{\alpha}\cdot\frac{1-\alpha}{1-\beta}\cdot K_1 \cdot L_2 \qquad \frac{(\alpha - \beta)^2}{\overline{K}.\overline{L}}\cdot K_1 \cdot L_2 \cdot \frac{d\overline{\theta}}{\overline{\theta}} \;=\; p_2 \;. \tag{43d}$$

Thus, p_2 is negatively related to changes in . In other words, the elasticity of P_2 with respect to is negative since

$$0 \;>\; -\,\frac{(\alpha - \beta)^2}{\overline{K}.\overline{L}}\cdot K_1 \cdot L_2 \;=\; \frac{dP_2}{d\overline{\theta}}.\frac{\overline{\theta}}{P_2} \qquad \forall\;\; \alpha\_\beta. \tag{43e}$$

Note that **d**mps is also *negatively* related to **d**θ since

$$d\overline{\theta} \;=\; -\;\frac{\gamma_1}{[1 - \gamma_1.(1 - mps)]^2}.dmps \;.$$

Thus, as *mps* increases, falls causing a rise in P_2. Then, *mps* is positively related to P_2, the price for the output of investment good producing sector. It is an easy matter now to show that *mps* is positively related also to PK_i for each i as

$$dPK_i \;=\; a_i 2 \cdot dP_2$$

implying that p_2=pk_i where pk_i=**d**ln PK_i=**d**PK_i/PK_i.

Having shown the comparative-statics effects of a rise in *mps* on P_2 and PK_i, we can now take a look at the effects on the use of resources by sectors. Remembering that P_1 is fixed, an increase in P_2 (following **d**mps>0) will imply a rise in the *relative* price of commodity 2 (in terms of commodity 1). In the context of a Walrasian model, under the assumption of perfect mobility of factors, this would result in a pull of resources towards sector 2. This is indeed the case since **d**L_2>0 and **d**K_2>0 after the rise in *mps*. To see this, consider the equations below.

$$dL_2 = -\frac{\alpha\beta\overline{L}}{(\alpha\overline{\theta}+\beta)^2}d\overline{\theta} = \frac{\alpha\beta\gamma_1\overline{L}}{(\alpha\overline{\theta}+\beta)^2[1-\gamma_1(1-mps)]^2}dmps$$

$$dK_2 = -\frac{(1-\alpha)(1-\beta)\overline{K}}{[(1-\alpha)\overline{\theta}+(1-\beta)]^2}d\overline{\theta} = \frac{(1-\alpha)(1-\beta)\gamma_1\overline{K}}{[(1-\alpha)\overline{\theta}+(1-\beta)]^2[1-\gamma_1(1-mps)]^2}dmps$$

The positive values of $\mathbf{d}L_2/\mathbf{d}mps$ and $\mathbf{d}K_2/\mathbf{d}mps$ imply, together with constant supplies of labor and capital (i.e., $d\bar{L}=d\bar{K}=0$), that $\mathbf{d}L_1/\mathbf{d}mps$ and $\mathbf{d}K_1/\mathbf{d}mps$ are both negative.

To summarize, therefore, an exogenous increase in *mps* leads to:

■ A rise in the prices paid for commodity 2 by both the final consumers and the producers who demand it for investment purposes as a capital good;

■ An increase in the output of commodity 2 --due to the increase in relative price of commodity 2 and the resulting resource pull by the sector, i.e., if $\mathbf{d}L_2>0$ and $\mathbf{d}K_2>0$, then, $\mathbf{d}X_2>0$ also; and

■ A fall in the output of commodity 1 --due to the fall in its relative price --if $\mathbf{d}L_2>0$ and $\mathbf{d}K_2>0$, then, $\mathbf{d}L_1<0$ and $\mathbf{d}K_1<0$ implying, in turn, that $\mathbf{d}X_2<0$.

As general equilibrium requires clearance of all markets, i.e., that supply be equal to demand, the second result above also points to a rise in the demand for commodity 2. This combined with the first result may seem surprising to those used to think in a Marshallian partial equilibrium framework, for it implies a *simultaneous* increase in both the own price of the commodity and the demand for it. One should note, however, that what is increasing together with price is *total* demand for commodity 2 (i.e., the sum of final consumption demand and investment demand) and that the final consumption component of the demand is still negatively related to price as shown in equation (30).

The Issue of Interest Rates

Since Walrasian models take money as neutral thereby ignoring money market adjustment in response to shocks, traditional CGE models in the Walrasian tradition (Derviş, de Melo, Robinson; Robinson) do not include an interest rate variable.[8] This affects negatively the

---

8 Recently, a new branch of literature incorporating a financial sector into the CGE models emerged. For a discussion on these models, see Robinson. For examples of such models, see Lewis; Díaz-Giménez *et.al.*, or Yeldan.

usefulness of the CGE framework in the analysis of the general equilibrium impact of macro shocks and often leads to *ad hoc* interpretations of simulation experiment results.[9]

The formulation suggested below allows for endogenous determination of an interest rate, IR, in a CGE model. Though it may appear to be a superfluous addition not affecting the fundamental structure of the simple model here,[10] it is useful in that it shows the implied change in IR that is consistent with the resulting relative price structure following an exogenous shock. In general, the incorporation of an endogenous interest rate formulation into a CGE model improves the informative capacity of the model in the empirical sense even without directly relating IR to any other endogenous variable.[11] To the extent that the interest rate defined here can be viewed as a proxy for the interest rates in the macroeconomic sense, the formulation may help establish a link between Walrasian and Keynesian models that are not necessarily compatible (e.g., Rogers; Sargent). It may thus prove to be useful especially in the analysis of the effects of macro shocks through large CGE models whose analytical solutions are impossible to find. To illustrate the idea, one can consider an area where the use of this formulation may increase the relevance of results from CGE simulation exercises: The analysis of the effects of budget deficits which has recently become the subject of a hot debate in the macroeconomics literature. In this context, the formulation provides a way of checking the validity of such arguments as in Adelman and Robinson whose analysis of U.S. macro imbalances (Footnote 9) is based on a model that does not allow for capital mobility.[12] Finally, it illustrates the complications that may arise in the comparative statics analysis of the effects of exogenous shocks --here, a change in

---

[9] Such interpretations about the real interest rate adjustment in the U.S. following the simultaneous growth of budget and current account deficits in the 1980s can be found, for example, in Adelman and Robinson.

[10] The formulation appears to be superfluous as it consists of the addition of an equation to determine a variab-le (IR) not used elsewhere in the model. It can be argued, therefore, that the suggested formulation lets the value of IR be determined but IR itself does not *directly* affect the value of any other endogenous variables. Ideally, some mechanism for IR to affect other variables (such as "foreign savings" in larger, open economy models) should be designed. This is not attempted here to keep the model analytically solvable. In addition, such attempts are argued to push the limits of Walrasian framework --see, for example, Robinson. As it is, the formulation adds a constraint upon values of capital rentals and capital goods prices without affecting the Walrasian nature of the model as all endogenous variables remain homogenous of degree 0 in all prices.

[11] It is better to have the extra information on the implied change in IR by simply adding the formulation to a model than not being able to obtain that information from the same model.

[12] For an empirical investigation for the U.S. economy and a more detailed discussion along these lines, see Sayan, Hushak and Tweeten.

mps. In particular, the direction of comparative statics effect on the interest rate can not be derived with certainty as discussed below.

The Formulation

$$IR = R / P_K \tag{44}$$

$$P_K = \varepsilon_1 \cdot PK_1 + \varepsilon_2 \cdot PK_2 \tag{45}$$

where P_K : aggregate price index for capital goods

ε_i : capital good price index weight factors ($\Sigma\varepsilon_i = 1$)

The system with equations (21)-(38) plus (44)-(45) has 20 equations (19 of which are independent) and 20 endogenous variables: X_i, L_i, K_i, W, R, Y, P_i, C_i, I_2, DK_i, PK_i, P_K and IR. We can again proceed by choosing commodity 1 as the numerairé so as to fix P_1.[13]

The formulation for the interest rate in (45) is based on the assumption of perfect mobility of capital between sectors.[14] This implies the existence of a perfect market in existing capital stock that is taken to substitute for a money market not incorporated into the model, as the existence of such a market justifies viewing the (durable) capital goods as stores of value thereby replacing money as a store of value. In this case, wealth and savings are defined respectively by the holdings and purchases of capital goods.

In this framework, the interest rate must be related to the prices of capital goods and the rental for capital (as a factor of production) so as to allow for it be determined endogenously. Considering capital goods as perfect substitutes for each other as stores of value, investors buy capital goods for their use as stores of value, not for their particular characteristics. Perfect substitutability of all capital goods in this sense guarantees that all savings will be invested in the capital good with the highest yield until all interest rates are equalized (Keller). Then, we must have

[13] Alternatively, we could have fixed P_K by letting P_K be the aggregate price index for capital goods and drop equation (45). Then, the system with equations (21)-(38) plus (54) would have 19 equations (18 of which are independent) and 19 endogenous variables: X_i, L_i, K_i, W, R, Y, P_i, C_i, I_2, DK_i, PK_i and IR. The equation (45) would have to be dropped because fixing P_K amounts to fixing the value of P_2 since the equation implies by (37) and (38)

$$P_2 = \frac{\overline{P}_K}{(\varepsilon_1 \cdot a_{21} + \varepsilon_2 \cdot a_{22})} .$$

[14] For a more realistic formulation taking into account the depreciation expenses on the existing capital stock and expected capital gains, see. Sayan, Hushak and Tweeten.

$$IR = \frac{R}{PK_1} \qquad \textbf{(45a)}$$

$$IR = \frac{R}{PK_2} \qquad \textbf{(45b)}$$

Multiplying both sides of (45a) by ε_1, both sides of (45b) by ε_2 and summing up the resulting equations yield equation (45). This equation can now be given the following interpretation: Since capital is both a factor of production and a store of value (representing accumulated savings of the past), the households owning the capital goods will be willing to rent the services of one unit of *composite* capital (worth \$ P_K) as a factor of production to the producers if the capital rental is at least as high as the *market* interest rate, IR.[15]

<u>The Solution Procedure</u>

As for the analytical solution of the system, we can write the downsized version in a similar way to what we did for the simpler Walrasian system, as follows:

$$\alpha . L_1^{(\alpha-1)} . K_1^{(1-\alpha)} = \beta . P_2 \cdot L_2^{(\beta-1)} . K_2^{(1-\beta)} \qquad \textbf{(48)}$$

$$(1-\alpha) . L_1^{\alpha} \cdot K_1^{-\alpha} = (1-\beta) . P_2 \cdot L_2^{\beta} \cdot K_2^{-\beta} \qquad \textbf{(49)}$$

$$\overline{L} = L_1 + L_2 \qquad \textbf{(50)}$$

$$\overline{K} = K_1 + K_2 \qquad \textbf{(51)}$$

$$\frac{L_1^{\alpha} \cdot K_1^{(1-\alpha)}}{L_2^{\beta} \cdot K_2^{(1-\beta)}} = \overline{\theta} . P_2 \qquad \textbf{(52)}$$

$$IR = \frac{(1-\alpha)}{(\varepsilon_1 \cdot a_{21} + \varepsilon_2 \cdot a_{22}) . P_2} L_1^{\alpha} \cdot K_1^{-\alpha} \qquad \textbf{(53)}$$

where $\bar{\theta}=\gamma_1(1\text{-mps})/[1-\gamma_1(1\text{-mps})]$. So, the downsized system (48) through (53) is comprised of 6 equations in 6 endogenous variables: L_i, K_i, IR and P_2. When solved for L_i, K_i and P_1, equations (48) through (52) give the same solution for L_i and K_i as in equations (19a) through (20b) except that θ now is given by $\gamma_1(1\text{-mps})$. Upon substitution of solution values

---

[15] In the microeconomic sense, IR is a "real" variable that is homogenous of °0 in R and PK_i. This is the reason why IR is often referred to as the *real interest rate* in the CGE literature. In macroeconomics, this term is used for nominal (stated) rate minus the rate of inflation. In this latter sense, IR in the equation is the nominal rate.

for L_i and K_i [from (19a) through (20b)] into (52), one can find the solution value for P_2. Substitution of this value into equation (53) yields IR.

<u>The Effect of a Change in "*mps*" on the Interest Rate</u>

Having obtained the solution values for endogenous variables, one can proceed with the investigation of the direction of the effect on the interest rate of a change in *mps*. To do this, we can first take the natural log of both sides of (53) to get

$$\ln IR = \ln \kappa - \ln P_2 + \alpha.\ln L_1 - \alpha.\ln K_1 \qquad \textbf{(54)}$$

where $\kappa = (1-\alpha)/(\varepsilon_1.a_{21}+\varepsilon_2.a_{22})$. Totally differentiating both sides, we write

$$ir = -p_2 + \alpha.(l_1 - k_1) \qquad \textbf{(55)}$$

where small case letters again indicate the percentage change in the variables denoted by respective capital letters. It can be shown using equations (19a)-(20b) that

$$l_1 \equiv \frac{dL_1}{L_1} = \frac{L_2}{\overline{L}}.\frac{d\overline{\theta}}{\overline{\theta}} \qquad \textbf{(56)}$$

$$k_1 \equiv \frac{dK_1}{K_1} = \frac{K_2}{\overline{K}}.\frac{d\overline{\theta}}{\overline{\theta}}. \qquad \textbf{(57)}$$

Substituting right hand side expressions from (43d) for p_2, (56) for l_1, and (57) for k_1 into (55) yields

$$ir = \left[\frac{(\alpha-\beta)^2}{\overline{K}.\overline{L}}.K_1.L_2 + \alpha.\left(\frac{L_2}{\overline{L}} - \frac{K_2}{\overline{K}}\right)\right].\frac{d\overline{\theta}}{\overline{\theta}} \qquad \textbf{(58)}$$

the right hand side of which is uncertain in sign. (58) can be rearranged into

$$ir = \frac{1}{\overline{K}.\overline{L}}.[(\alpha-\beta)^2.K_1.L_2 + \alpha.(K_1.L_2 - K_2.L_1)].\frac{d\overline{\theta}}{\overline{\theta}} \qquad \textbf{(58a)}$$

or, equivalently,

$$ir = \frac{K_1.L_2}{\overline{K}.\overline{L}}.\left[(\alpha-\beta)^2 + \frac{\alpha.(\beta-\alpha)}{\beta.(1-\alpha)}\right].\frac{d\overline{\theta}}{\overline{\theta}} \qquad \textbf{(58b)}$$

where use is made of the fact that

$$K_1.L_2 = \frac{\beta}{\alpha}.\frac{1-\alpha}{1-\beta}.K_2.L_1 \qquad \textbf{(59)}$$

$$ir = \frac{K_1.L_2}{\overline{K}.\overline{L}}.(\alpha-\beta)^2.\left[1 - \frac{\alpha}{\beta}.\frac{1}{(1-\alpha)}.\frac{1}{(\alpha-\beta)}\right].\frac{d\overline{\theta}}{\overline{\theta}} \qquad \textbf{(60)}$$

the right hand side of which is still uncertain in sign because of the uncertainty about the sign of the term in brackets. Let's consider two possibilities i) $\alpha<\beta$, and ii) $\alpha>\beta$.

Under (i), the term in brackets will be positive implying that IR is positively related to . In other words, a rise in *mps* would lead to a fall in IR (since falls as *mps* increases, as pointed out earlier). Under (ii), on the other hand, the term in brackets will be negative since then each of the parametric terms in brackets would be greater than 1 (i.e., $\alpha/ß>1$, $1/(1-\alpha)>1$, and $1/(\alpha-ß)>1$) making their product also exceed 1. Under these circumstances, an increase in would lead to a fall in IR or equivalently, a rise in *mps* would lead to a rise in IR.

The comparative statics analysis, therefore, shows that whether an increase in *mps* causes the IR to go up or down depends on the relative magnitudes of α and ß. We take a closer look at the nature of this relationship below.[16]

The Relationship between "α" and "ß"

Proposition:

$\alpha>ß$ ($\alpha<ß$) iff labor intensity of production in sector 1 is greater (smaller) than that in sector 2. Equivalently, $\alpha>ß$ ($\alpha<ß$) iff production in sector 2 is more (less) capital intensive than that in sector 1.

Proof:

From (39) and (40), we write

$$\frac{\alpha}{(1-\alpha)}\frac{K_1}{L_1} = \frac{\beta}{(1-\beta)}.\frac{K_2}{L_2} \tag{61}$$

or equivalently,

$$\frac{L_1}{K_1} = \frac{\alpha}{(1-\alpha)}.\frac{(1-\beta)}{\beta}.\frac{L_2}{K_2}\ . \tag{62}$$

If $L_1/K_1 > L_2/K_2$ (i.e., labor intensity in sector 1 exceeds that in sector 2), then we must have

$$\frac{\alpha}{(1-\alpha)}.\frac{(1-\beta)}{\beta} > 1 \tag{63}$$

implying that $\alpha>ß$.

[16] See Devarajan and Offerdal. Though the analysis there has entirely different purposes, their comparative statics results also depend on the relative capital intensities (capital-labor ratios) of sectoral production functions.

Concluding Remarks

The discussion above indicates that it is possible to define an interest rate variable that can be incorporated into a Walrasian CGE model with no financial sector. The incorporation of such a variable does not appear to affect the fundamental structure of the model. Nonetheless, it remains useful for the following reasons. Firstly, it shows the implied change in IR that is consistent with the resulting relative price structure following an exogenous shock. Generalizing the argument, the incorporation of an endogenous interest rate formulation into a CGE model improves the informative capacity of the model in the empirical sense even when IR is not directly related to any other endogenous variable. Secondly, to the extent that the interest rate defined can be viewed as a proxy for the interest rates in the macroeconomic sense, the formulation may help establish a link between Walrasian and Keynesian models that are not necessarily compatible. It may thus prove to be especially useful in the applied analysis of the effects of macro shocks. To illustrate the idea, one can consider an area where the use of this formulation may increase the relevance of results from CGE simulation exercises: The analysis of the effects of budget deficits which has recently become the subject of a hot debate in the macroeconomics literature. The numerical techniques may be used to derive the direction of possible effects on IR of a change in, say, the budget position of the government. Hence, when the interest rate formulation is included, CGE analysis enables one to take a closer look at the sectoral effects of macroeconomic imbalances associated with a rise or a fall in the interest rate. Thus, utilization of the existing potential of CGE analysis to shed some light on such a dim area of the literature via what amounts to a relatively small modification in the model structure seems like a good idea.

Another justification for the inclusion of an interest rate formulation is that such a formulation can provide additional insights into the role of the sectoral production technologies (or the assumptions concerning these technologies) in transmission of the effects of exogenous shocks throughout the economy. This is especially important in the context of a multisectoral model where sector aggregation issues present additional difficulties.[17]

[17] In such a model, output of each sector may be usable both as a final consumption good and as an investment good. The share of the part of output usable for final consumption varies from sector to sector, however, depending upon the aggregation scheme used.

BIBLIOGRAPHY

Adelman, I., and S. Robinson (1988). "Macroeconomic Shocks, Foreign Trade, and Structural Adjustment: A General Equilibrium Analysis of the U.S. Economy, 1982-1986." in ***Elasticities in International Agricultural Trade***, ed. C.Carter and W.H. Gardiner, pp. 137-62. Boulder CO: Westview Press.

Bolnick, B.R. (1989). ***The ABCs of CGEs: Computable General Equilibrium Models for Development Planning***. Development Discussion Paper No. 326. Cambridge, MA: Harvard Institute for International Development.

Derviş, K., J. de Melo and S. Robinson (1982). ***General Equilibrium Models for Development Policy***. Cambridge, U.K.: Cambridge University Press.

Devarajan, S., and E. Offerdal (1989). "Capital Markets and Computable General Equilibrium Models: Comparative Statics without Apology?" ***Journal of Policy Modeling***. 11: 191-212.

Díaz-Giménez, J. *et.al.* (1992). ***Banking in Computable General Equilibrium Economies*** Federal Reserve Bank of Minneapolis Research Department, Staff Report No. 153.

Keller, W. J. (1980). ***Tax Incidence: A General Equilibrium Approach***. Amsterdam, Netherlands: North Holland, 1980.

Lewis, J.D. (1985). ***Financial Liberalization and Price Rigidities in a General Equilibrium Model with Financial Markets***. Development Discussion Paper No. 211. Cambridge, MA: Harvard Institute for International Development.

Robinson, S. (1989). "Multisectoral Models." in ***Handbook of Development Economics***, ed. H. Chenery and T.N. Srinivasan, Volume II, pp. 885-947. Amsterdam, Netherlands: North Holland.

Robinson, S., M. Kilkenny and K. Hanson (1990). ***The USDA/ERS Computable General Equilibrium (CGE) Model of the United States***. Washington, DC: U.S. Department of Agriculture, Agriculture and Rural Economy Division, Economic Research Service, Staff Report No. AGES9049.

Rogers, C.(1989). ***Money, Interest and Capital A Study In the Foundations of MonetaryTheory***, Cambridge: Cambridge University Press.

Sargent, T.J. (1979). ***Macroeconomic Theory***, New York: Academic Press.

Sayan, S., L.J. Hushak and L. G. Tweeten (1993). "A Computable General Equilibrium Analysis of the Effects of U.S. Macroeconomic Imbalances under Perfect Capital Mobility." Bilkent University, Ankara, Turkey.

Yeldan, E. (1992). ***Financial Liberalization and Fiscal Repression in Turkey: Policy Analysis in a CGE Model with Financial Markets***. Discussion Paper No. 92-7. Bilkent University, Ankara, Turkey.